NORMAL FORMS FOR ORTHOGONAL SIMILARITY CLASSES OF SKEW-SYMMETRIC MATRICES

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ABSTRACT. Let F be an algebraically closed field of characteristic different from 2. Define the orthogonal group, $O_n(F)$, as the group of n by n matrices X over F such that $XX' = I_n$, where X' is the transpose of X and I_n the identity matrix. We show that every nonsingular n by n skew-symmetric matrix over F is orthogonally similar to a bidiagonal skew-symmetric matrix. In the singular case one has to allow some 4-diagonal blocks as well.

If further the characteristic is 0, we construct the normal form for the $O_n(F)$-similarity classes of skew-symmetric matrices. In this case, the known normal forms (as presented in the well known book by Gantmacher) are quite different.

Finally we study some related varieties of matrices. We prove that the variety of normalized nilpotent n by n bidiagonal matrices for $n = 2s + 1$ is irreducible of dimension s. As a consequence the skew-symmetric nilpotent n by n bidiagonal matrices are shown to form a variety of pure dimension s.

1. INTRODUCTION

In this note F denotes an algebraically closed field of characteristic not 2. By $M_n(F)$ we denote the algebra of n by n matrices over F, by $GL_n(F)$ the group of invertible elements of $M_n(F)$, and by I_n the identity matrix of size n. For any matrix X, let X' denote the transpose of X. The orthogonal group, $O_n(F)$, is defined as the subgroup of $GL_n(F)$ consisting of matrices X such that $XX' = I_n$.

This paper is a sequel to [1] where the first and third author constructed tridiagonal normal forms for symmetric matrices under the action of $O_n(F)$. Here we continue this work and study the similarity action of $O_n(F)$ on the space $\text{Skew}_n(F)$ of all n by n skew-symmetric matrices over F. We show that each $A \in \text{Skew}_n(F)$ is orthogonally similar to the direct sum of blocks B which are either bidiagonal or 4-diagonal. The latter is needed only if A has at least one pair of nilpotent Jordan blocks of even size. Each of the blocks B is similar to either a pair of Jordan blocks having the same size, s, and having eigenvalues λ and $-\lambda$, with s even if $\lambda = 0$, or a single nilpotent Jordan block of odd size.

If F has characteristic 0, we are able to choose the concrete representatives, i.e., normal forms for skew-symmetric matrices, see Theorem 3.2. In this case, the known normal forms for the $O_n(F)$-similarity classes of nonsingular skew-symmetric

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matrices, as presented in the well known book by Gantmacher [2], are not bidiagonal. Hence our new normal forms are simpler and may be better suited for some applications. At the end of Section 3 we pose three open problems. The proof of Lemma 3.1 uses some ideas of Givental and, due to its length, is given separately in Section 4.

In the last section we study the variety $\mathcal{B}_n \cap \mathcal{N}_n$ of nilpotent n by n bidiagonal matrices with 1's along the lower diagonal (normalized nilpotent bidiagonal matrices), where $n = 2s + 1$. We compare $\mathcal{B}_n \cap \mathcal{N}_n$ with the variety of normalized nilpotent tridiagonal matrices $\mathcal{T}_n \cap \mathcal{N}_n$ studied by Kostant in relation with the Toda lattice [4], and we show that $\mathcal{B}_n \cap \mathcal{N}_n$ is irreducible of dimension s. These results also have consequences for the related variety of skew-symmetric nilpotent bidiagonal matrices, see Corollary 5.5. Finally we note that the coordinate ring of $\mathcal{B}_n \cap \mathcal{N}_n$ has an interpretation in terms of quantum cohomology of the flag variety SL_n/B.

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2. Bidiagonal and 4-diagonal representatives

We first recall the following well known facts (see e.g. [6, Theorem 70] and [2, Chapter XI, Theorem 7]). Gantmacher gives the proof over complex numbers but his argument is valid for any algebraically closed field of characteristic different from 2.

Theorem 2.1. *If two skew-symmetric matrices $A, B \in M_n(F)$ are similar, then they are orthogonally similar.*

Let t be an indeterminate over F and let i denote one of the two square roots of -1 in F. Thus $i^2 = -1$.

Theorem 2.2. *A matrix $A \in M_n(F)$ is similar to a skew-symmetric matrix iff the elementary divisors $(t - \lambda)^s$ and $(t + \lambda)^s$ of A come in pairs if $\lambda \neq 0$ or s is even.*

We are interested in constructing normal forms of skew-symmetric matrices under orthogonal similarity. The case of complex matrices is classical and is described in Gantmacher's book [2]. Our objective is to construct a normal form which is almost bidiagonal, unlike the one given by Gantmacher.

The general form of a skew-symmetric bidiagonal matrix is

$$
S = \begin{bmatrix}
0 & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-a_1 & 0 & a_2 & 0 & & 0 & 0 & 0 \\
0 & -a_2 & 0 & a_3 & & 0 & 0 & 0 \\
0 & 0 & -a_3 & 0 & & 0 & 0 & 0 \\
\vdots & & & & & & & \\
0 & 0 & 0 & 0 & & 0 & a_{n-2} & 0 \\
0 & 0 & 0 & 0 & & -a_{n-2} & 0 & a_{n-1} \\
0 & 0 & 0 & 0 & & 0 & -a_{n-1} & 0
\end{bmatrix}.
$$

It is easy to see that if all a_k's are nonzero, then S is cyclic (i.e., its minimal and characteristic polynomials coincide). If all $a_k \neq 0$, and $a_k = 1$ for even k's then we say that this bidiagonal matrix is *special*.

Let s be a positive integer. We introduce a skew-symmetric nilpotent matrix Q_{4s} of size $4s$ having the elementary divisors t^{2s}, t^{2s}. The easiest way to understand the structure of this matrix is to take a look at one example:

$$
Q_{12} = \begin{bmatrix}
0 & \boxed{-1 \quad i} & & & & & & & & & & 0 \\
1 & 0 & 0 & & & & & & & & & \\
-i & 0 & 0 & \boxed{-1 \quad i} & & & & & & & & \\
& & 1 & 0 & 0 & & & & & & & \\
& & -i & 0 & 0 & \boxed{-1 \quad i} & & & & & \\
& & & & 1 & 0 & 0 & -i & & & \\
& & & & -i & 0 & 0 & -1 & & & \\
& & & & \boxed{i \quad 1} & 0 & 0 & -i & & \\
& & & & & & 0 & 0 & -1 & & \\
& & & & & & \boxed{i \quad 1} & 0 & 0 & -i \\
& & & & & & & & 0 & 0 & -1 \\
0 & & & & & & & & \boxed{i \quad 1} & 0
\end{bmatrix}
$$

In order to define precisely these matrices, let us write $E_{p,q}$ for a square matrix (of appropriate size, $4s$ in this case) all of whose entries are 0 except the (p,q)-entry which is 1. Then we have

$$Q_{4s} = X - X',$$

where $X = X_1 + X_2$ and

$$
X_1 \;=\; \sum_{p=1}^{s} \left(iE_{2p-1,2p+1} - E_{2p-1,2p} \right),
$$

$$
X_2 \;=\; \sum_{p=s+1}^{2s} \left(iE_{2p,2p-2} + E_{2p,2p-1} \right).
$$

In the above example, the nonzero entries of X are enclosed in small boxes.

We shall now prove the above mentioned properties.

Lemma 2.3. *The skew-symmetric matrix Q_{4s} is nilpotent and has elementary divisors t^{2s}, t^{2s}.*

Proof. It is easy to verify that $XX' = 0$, and also that $X_1 X_2 = X_2 X_1 = 0$ and $X_1^{s+1} = X_2^{s+1} = 0$. The matrix X_1^s resp. X_2^s has exactly two nonzero entries namely i^{s+1}, i^s resp. i^s, i^{s-1} and they are positioned in the middle of the first resp. last row. It follows easily that $X^{s+1} = 0$, $X^s = X_1^s + X_2^s$, and $X'^s X^s = 0$. By using these relations and $XX' = 0$, we obtain that

$$
\begin{aligned}
Q_{4s}^{2s} &= (X - X')^{2s} \\
&= X^{2s} - X'X^{2s-1} + X'^2 X^{2s-2} - \cdots - X'^{2s-1}X + X'^{2s} \\
&= (-1)^s X'^s X^s = 0.
\end{aligned}
$$

As Q_{4s} is nilpotent of rank $4s - 2$, it follows that it has exactly two Jordan blocks each of size $2s$. $\qquad\square$

Remark 2.4. The matrix Q_{4s} in the above lemma cannot be replaced by a bidiagonal skew-symmetric matrix S (of size $n = 4s$). Indeed assume that S is the bidiagonal matrix displayed above with superdiagonal entries $a_1, ..., a_{n-1}$. Since S must have rank $n - 2$, exactly one of the a_k's is 0. As it has elementary divisors t^{2s}, t^{2s} we

conclude that $a_{2s} = 0$. This contradicts Theorem 2.2 as the $2s \times 2s$ block in the upper left hand corner has size $2s$ and only one elementary divisor t^{2s}.

We can now state and prove one of our main results.

Theorem 2.5. *Every skew-symmetric matrix $A \in M_n(F)$ is orthogonally similar to the direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_m$, where each block A_k is one of the following:*

(a) special bidiagonal of size $2s$ with the elementary divisors $(t - \lambda)^s, (t + \lambda)^s$ and $\lambda \neq 0$;

(b) special bidiagonal of size $2s + 1$ with the elementary divisor t^{2s+1};

(c) Q_{4s}, with the elementary divisors t^{2s}, t^{2s}.

Proof. In view of Theorems 2.1 and 2.2, it suffices to construct $A \in \mathrm{Skew}_n(F)$ having only one or two elementary divisors, as specified in the three cases of the theorem.

Let us start with the case (a). Then $n = 2s$ and the two elementary divisors are $(t-\lambda)^s$ and $(t+\lambda)^s$ with $\lambda \neq 0$. Let $S = S(x_1, x_2, \ldots, x_s, y)$ be the skew-symmetric bidiagonal matrix whose entries on the first superdiagonal are the indeterminates

$$x_1, y, x_2, y, \ldots, x_{s-1}, y, x_s$$

in this order.

By using an obvious permutation matrix P, we can transform S^2 to obtain direct sum of two matrices of size s each:

$$PS^2P^{-1} = S_1 \oplus S_2.$$

The matrix S_1 resp. S_2 is the submatrix of S^2 which lies in the intersection of rows and columns having odd resp. even indices. These two matrices are tridiagonal and symmetric. Explicitly, the entries on the first superdiagonal of S_1 are

$$x_1 y, x_2 y, \ldots, x_{s-1} y;$$

and its diagonal entries are

$$-x_1^2, -x_2^2 - y^2, \ldots, -x_{s-1}^2 - y^2, -x_s^2 - y^2.$$

The corresponding entries of S_2 are

$$x_2 y, x_3 y, \ldots, x_s y;$$

and

$$-x_1^2 - y^2, -x_2^2 - y^2, \ldots, -x_{s-1}^2 - y^2, -x_s^2.$$

We claim that the matrices $S_1 = S_1(x_1, \ldots, x_s, y)$ and $S_2 = S_2(x_1, \ldots, x_s, y)$ have the same characteristic polynomial. Let X be the $s \times s$ matrix whose diagonal entries are $x_1, x_2, \ldots, x_s$, those on the first subdiagonal are all equal to $-y$, while all other entries are 0. Then $S_1 = -XX'$ and $S_2 = -X'X$ and our claim follows.

Let us write the characteristic polynomial of the matrix $\lambda^2 y^2 I_s - S_1(x_1, \ldots, x_s, y)$ as

$$p(t) = t^s + c_1 t^{s-1} + \cdots + c_{s-1}t + c_s,$$

where $c_k = c_k(x_1, \ldots, x_s, y)$ is a homogeneous polynomial of degree $2k$ in the indicated variables.

Since there are $s+1$ indeterminates and F is algebraically closed, by [7, Theorem 4, Corollary 5, p.57] the system of homogeneous equations:

$$c_k(x_1, x_2, \ldots, x_s, y) = 0, \quad k = 1, 2, \ldots, s$$

has a nontrivial solution in F^{s+1}, say $(\xi_1, \xi_2, \ldots, \xi_s, \eta)$. It follows that the matrix

$$\lambda^2 \eta^2 I_n - S(\xi_1, \ldots, \xi_s, \eta)^2$$

is nilpotent. We have $\eta \neq 0$ since otherwise this matrix would be diagonal and nilpotent, i.e., zero. The matrix

$$A = \eta^{-1} S(\xi_1, \ldots, \xi_s, \eta)$$

is bidiagonal and the matrix $\lambda^2 I_n - A^2$ nilpotent. As A is skew-symmetric, its eigenvalues must be λ and $-\lambda$, each with multiplicity s.

Next we claim that A is special, i.e., that all $\xi_k \neq 0$. We shall prove this claim by contradiction. So, assume that $\xi_k = 0$ for some k. Then the matrix $\lambda^2 \eta^2 I_s - S_1(\xi_1, \cdots, \xi_s, \eta)$ breaks into direct sum of two blocks: the first X_1 of size k and the second of size $s - k$ (if $k = s$ the second block is of size 0). Similarly, the matrix $\lambda^2 \eta^2 I_s - S_2(\xi_1, \cdots, \xi_s, \eta)$ breaks into direct sum of two blocks: the first X_2 of size $k - 1$ and the second of size $s - k + 1$ (if $k = 1$ the first block is of size 0). Since all these four blocks are nilpotent, their traces must be 0. Since

$$\begin{aligned}
\text{tr } (X_1) &= k\lambda^2 \eta^2 + (k-1)\eta^2 + (\xi_1^2 + \cdots + \xi_{k-1}^2), \\
\text{tr } (X_2) &= (k-1)\lambda^2 \eta^2 + (k-1)\eta^2 + (\xi_1^2 + \cdots + \xi_{k-1}^2),
\end{aligned}$$

we deduce that $\lambda^2 \eta^2 = 0$, which is a contradiction.

Hence the matrix $\lambda I_n - A$ has rank $n - 1$. Consequently the elementary divisors of A are indeed $(t - \lambda)^s$ and $(t + \lambda)^s$.

The case (b) will be handled by a similar argument. Let $S = S(x_1, x_2, \ldots, x_s, y)$ be the skew-symmetric bidiagonal matrix whose entries on the first superdiagonal are the indeterminates

$$x_1, y, x_2, y, \ldots, x_{s-1}, y, x_s, y$$

in this order. The characteristic polynomial of this matrix has the form

$$p(t) = t^{2s+1} + c_1 t^{2s-1} + \cdots + c_{s-1} t^3 + c_s t,$$

where $c_k = c_k(x_1, \ldots, x_s, y)$ is a homogeneous polynomial of degree $2k$ in the indicated variables.

Since there are $s + 1$ indeterminates and F is algebraically closed, the system of homogeneous equations:

$$c_k(x_1, x_2, \ldots, x_s, y) = 0, \quad k = 1, 2, \ldots, s$$

has a nontrivial solution in F^{s+1}, say $(\xi_1, \xi_2, \ldots, \xi_s, \eta)$. It follows that the matrix $S(\xi_1, \ldots, \xi_s, \eta)$ is nilpotent. We claim that $\eta \neq 0$. Otherwise this matrix would be nilpotent and semisimple, i.e., the zero matrix.

Moreover we claim that each $\xi_k \neq 0$. We prove this by contradiction. Thus assume that some $\xi_k = 0$. Then the above matrix breaks up into direct sum of two blocks: the first of size $2k - 1$ and the second of size $2(s - k + 1)$. Both of these blocks must be nilpotent. However, the determinant of the second block is $\eta^{2(s-k+1)} \neq 0$, which gives a contradiction.

Hence the matrix $A = \eta^{-1} S(\xi_1, \ldots, \xi_s, \eta)$ is a special bidiagonal matrix having only one elementary divisor, t^{2s+1}.

The case (c) is handled by the above lemma.

$\square$

Next we show that there are only finitely many special bidiagonal matrices in $\text{Skew}_n(F)$ having prescribed elementary divisors.

Theorem 2.6. *If $n = 2s$ is even and $\lambda \neq 0$, then there are at most $2^s s!$ special bidiagonal matrices $A \in \mathrm{Skew}_n(F)$ with elementary divisors $(t - \lambda)^s$ and $(t + \lambda)^s$. If $n = 2s + 1$ is odd, then there are at most $2^s s!$ special bidiagonal matrices $A \in \mathrm{Skew}_n(F)$ with the elementary divisor t^{2s+1}.*

Proof. The existence of such matrices was proved in the previous theorem. We just have to show that the system of s homogeneous polynomial equations $c_k = c_k(x_1, \ldots, x_s, y) = 0$ in $s + 1$ variables has at most $2^s s!$ solutions in the associated projective space.

We claim that the number of solutions is finite. Otherwise the projective variety defined by this system of equations would possess an irreducible component, say X, of positive dimension. Consequently, the intersection of X with the hyperplane $y = 0$ would be non-empty. On the other hand we have shown in the proof of the previous theorem that there are no nontrivial solutions with $y = 0$. Thus our claim is proved.

Now the assertion of the theorem follows from Bézout's Theorem (see [7, Chapter IV, §2]). $\qquad\square$

The non-uniqueness of special bidiagonal matrices in $\mathrm{Skew}_n(F)$, with specified elementary divisors, prevents us from obtaining a genuine normal form.

3. Normal forms in characteristic zero

From now on we restrict our attention to algebraically closed fields F of characteristic 0. In this case we find very simple normal forms for orthogonal similarity classes of skew-symmetric matrices. In the case of nonsingular matrices, this normal form is bidiagonal.

Let $n = 2s + 1$ be odd and let $R_n \in \mathrm{Skew}_n(F)$ be the bidiagonal matrix whose consecutive superdiagonal entries are

$$\sqrt{s}, \; i, \; \sqrt{s - 1}, \; i\sqrt{2}, \; \sqrt{s - 2}, \; i\sqrt{3}, \; \ldots, \; \sqrt{3}, \; i\sqrt{s - 2}, \; \sqrt{2}, \; i\sqrt{s - 1}, \; 1, \; i\sqrt{s}.$$

For instance,

$$R_7 = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & i & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & i\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & -i\sqrt{3} & 0 \end{bmatrix}.$$

It is not hard to show that the matrix R_n is nilpotent. The proof is similar to the proof of [1, Proposition 3.3] and we shall omit it.

For even $n = 2s$, let $P_n \in \mathrm{Skew}_n(F)$ be the bidiagonal matrix whose consecutive superdiagonal entries are

$$\alpha_k = \sqrt{\frac{k(n - k)}{(n - 2k)^2 - 1}}, \quad 1 \leq k \leq n - 1.$$

Note that for $k \neq s$ the number α_k^2 is a positive rational number while $\alpha_s^2 = -s^2$. We may assume that $\alpha_s = si$. For instance,

$$P_6 = \frac{1}{3} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 2\sqrt{6} & 0 & 0 & 0 \\ 0 & -2\sqrt{6} & 0 & 9i & 0 & 0 \\ 0 & 0 & -9i & 0 & 2\sqrt{6} & 0 \\ 0 & 0 & 0 & -2\sqrt{6} & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}.$$

Lemma 3.1. *Let $n = 2s$ be even and let P_n be the matrix defined above. Then $(P_n^2 - I_n)^s = 0$.*

The proof of this lemma is somewhat long and complicated and will be given in the next section.

Since one has to extract square roots, there is a built in non-uniqueness in the definition of the matrices P_n and R_n. Hence they should be viewed as defined only up to the choice of these square roots, or equivalently, up to the action of the group of diagonal matrices with the diagonal entries ± 1. By abusing the language, we shall refer to this type of non-uniqueness as the *choice of signs*.

Since all entries on the superdiagonal of P_n are nonzero, we conclude that the elementary divisors of P_n are $(t - 1)^s$ and $(t + 1)^s$.

As a consequence, we obtain the following theorem.

Theorem 3.2. *Let F be an algebraically closed field of characteristic 0. Then any skew-symmetric matrix $A \in M_n(F)$ is orthogonally similar to the direct sum of blocks of the following types:*
   (a) λP_m, m even, $\lambda \neq 0$;
   (b) Q_m, m divisible by 4;
   (c) R_m, m odd.
   This direct decomposition is unique up to the ordering of the diagonal blocks and the choice of signs inside the blocks of type P_m and R_m.

We can now derive some interesting combinatorial identities from Lemma 3.1. We fix a positive integer s, set $n = 2s$, and define the coefficients

$$(3.1) \qquad \beta_k = \alpha_k^2 = \frac{k(n-k)}{(n-2k)^2 - 1}, \quad k \in \mathbf{Z}.$$

Note that then $\beta_k = \beta_{n-k}$ is valid for all integers k.

Corollary 3.3. *For $1 \leq k \leq s$ we have*

$$\sum_{1 \leq i_1 \ll i_2 \ll \cdots \ll i_{k-1} \ll i_k \leq n-1} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_{k-1}} \beta_{i_k} = (-1)^k \binom{s}{k},$$

where $i \ll j$ means that $j - i \geq 2$.

Proof. The characteristic polynomial of the matrix P_n is

$$f(t) = t^n - c_1 t^{n-2} + c_2 t^{n-4} - \cdots + (-1)^s c_s,$$

where

$$c_k = \sum_{1 \leq i_1 \ll i_2 \ll \cdots \ll i_{k-1} \ll i_k \leq n-1} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_{k-1}} \beta_{i_k}.$$

By Lemma 3.1, P_n has elementary divisors $(t-1)^s$, $(t+1)^s$. Hence, $f(t) = (t^2-1)^s$ and the identities stated in the corollary follow. $\square$

We end this section by proposing three related open problems.

Problem 1 Find a direct proof of the combinatorial identities stated in the above corollary.

Problem 2 If F has characteristic zero and $n = 2s$ is even, then there are exactly $2^s s!$ special bidiagonal matrices in $\mathrm{Skew}_n(F)$ having elementary divisors $(t-1)^s$, $(t+1)^s$. (We have verified this claim for $s \leq 4$.)

Problem 3 If F has characteristic zero and $n = 2s+1$ is odd, then there are exactly $2^s s!$ nilpotent special bidiagonal matrices in $\mathrm{Skew}_n(F)$. (We have verified this claim for $s \leq 4$.)

4. Proof of Lemma 3.1

We recall that $n = 2s$ is even and that the β_k's are defined by the formula (3.1). Clearly, the matrix P_n is similar to

$$
X = \begin{bmatrix}
0 & \beta_1 & 0 & & 0 & 0 \\
-1 & 0 & \beta_2 & & 0 & 0 \\
0 & -1 & 0 & & 0 & 0 \\
& & & \ddots & & \\
0 & 0 & 0 & & 0 & \beta_{n-1} \\
0 & 0 & 0 & & -1 & 0
\end{bmatrix}.
$$

The matrix X^2 has zero entries in positions (i,j) with $i+j$ odd. Hence X^2 is permutationally similar to the direct sum of two $s \times s$ matrices Y and Z. The matrix Y (resp. Z) is the submatrix of X^2 occupying the entries in positions (i,j) with i and j odd (resp. even). The matrices Y and Z are similar since $Y = UV$ and $Z = VU$, where

$$
U = \begin{bmatrix}
\beta_1 & 0 & 0 & & 0 & 0 \\
-1 & \beta_3 & 0 & & 0 & 0 \\
0 & -1 & \beta_5 & & 0 & 0 \\
& & & \ddots & & \\
0 & 0 & 0 & & \beta_{n-3} & 0 \\
0 & 0 & 0 & & -1 & \beta_{n-1}
\end{bmatrix},
$$

and

$$
V = \begin{bmatrix}
-1 & \beta_2 & 0 & & 0 & 0 \\
0 & -1 & \beta_4 & & 0 & 0 \\
0 & 0 & -1 & & 0 & 0 \\
& & & \ddots & & \\
0 & 0 & 0 & & -1 & \beta_{n-2} \\
0 & 0 & 0 & & 0 & -1
\end{bmatrix}.
$$

It remains to prove that the matrix Y is unipotent, i.e., the matrix $I_s - Y$ is nilpotent. For the reader's convenience, let us display the matrix $I_s - Y$:

$$
\begin{bmatrix}
1+\beta_1 & -\beta_1\beta_2 & 0 & 0 & 0 \\
-1 & 1+\beta_2+\beta_3 & -\beta_3\beta_4 & 0 & 0 \\
0 & -1 & 1+\beta_4+\beta_5 & 0 & 0 \\
& & \ddots & & \\
0 & 0 & 0 & 1+\beta_{n-4}+\beta_{n-3} & -\beta_{n-3}\beta_{n-2} \\
0 & 0 & 0 & -1 & 1+\beta_{n-2}+\beta_{n-1}
\end{bmatrix}.
$$

The proof of this last fact is quite intricate, it uses Givental's proof of nilpotency of some special tridiagonal matrices constructed from a finite quiver (see his paper [3]). For the reader's convenience, we include more detailed proof.

Denote by Γ the infinite square grid with vertex set $\mathbf{Z}^2$ and orient each horizontal edge to the right, $(i,j) \to (i+1,j)$, and each vertical edge downward, $(i,j+1) \to (i,j)$. To this horizontal resp. vertical edge we assign the weight

$$
u_{i,j} = -\frac{2i(2i+1)}{(2i-2j+1)(2i-2j+3)}
$$

resp.

$$
v_{i,j} = \frac{2j(2j-1)}{(2i-2j-1)(2i-2j+1)}.
$$

It is easy to verify that

$$(4.1) \qquad u_{i,j} + v_{i,j-1} = u_{i-1,j} + v_{i,j}, \quad u_{i,j}v_{i,j} = u_{i,j+1}v_{i+1,j},$$

i.e., for each vertex $(i,j) \in \Gamma$ the sum of the weights of the two incoming edges is the same as for the two outgoing edges and, for each small square of Γ, the product of the edge weights along the two oriented paths of length 2 are equal (see Figure 1). Note that $u_{0,j} = 0$ for all j's and $v_{i,0} = 0$ for all i's.
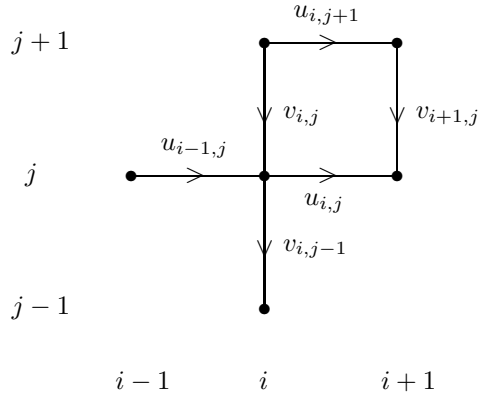

Fig. 1: A portion of Γ

For each integer $d \geq 1$ define two square matrices of size $d+1$

$$U_d = \begin{bmatrix} u_{d,1} & 0 & 0 & & 0 & 0 \\ 1 & u_{d-1,2} & 0 & & 0 & 0 \\ 0 & 1 & u_{d-2,3} & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & & u_{1,d} & 0 \\ 0 & 0 & 0 & & 1 & 0 \end{bmatrix},$$

and

$$V_d = \begin{bmatrix} -1 & v_{d,1} & 0 & & 0 & 0 \\ 0 & -1 & v_{d-1,2} & & 0 & 0 \\ 0 & 0 & -1 & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & & -1 & v_{1,d} \\ 0 & 0 & 0 & & 0 & -1 \end{bmatrix}.$$

The matrices

$$U_d V_d = \begin{bmatrix} -u_{d,1} & u_{d,1}v_{d,1} & 0 & & 0 & 0 \\ -1 & v_{d,1} - u_{d-1,2} & u_{d-1,2}v_{d-1,2} & & 0 & 0 \\ 0 & -1 & v_{d-1,2} - u_{d-2,3} & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & & v_{2,d-1} - u_{1,d} & u_{1,d}v_{1,d} \\ 0 & 0 & 0 & & -1 & v_{1,d} \end{bmatrix}$$

and

$$V_d U_d = \begin{bmatrix} v_{d,1} - u_{d,1} & u_{d-1,2}v_{d,1} & 0 & & 0 & 0 \\ -1 & v_{d-1,2} - u_{d-1,2} & u_{d-2,3}v_{d-1,2} & & 0 & 0 \\ 0 & -1 & v_{d-2,3} - u_{d-2,3} & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & & v_{1,d} - u_{1,d} & 0 \\ 0 & 0 & 0 & & -1 & 0 \end{bmatrix}$$

are tridiagonal.

One uses induction on $d \geq 1$ to prove that $U_d V_d$ is nilpotent. The case $d = 1$ is trivial to verify. Now let $d > 1$. As $U_d V_d$ and $V_d U_d$ are similar, it suffices to prove that $V_d U_d$ is nilpotent. This is the case iff the submatrix obtained from $V_d U_d$ by deleting the last row and column is nilpotent. The equalities (4.1) imply that this submatrix is equal to $U_{d-1}V_{d-1}$. Hence we have shown that all matrices $U_d V_d$ are nilpotent.

Finally it remains to observe that $I_s - Y = U_{s-1}V_{s-1}$, which can be easily verified by using the definition of the weights $u_{i,j}$ and $v_{i,j}$.

5. Some related varieties of matrices

Let $\mathcal{N}_n$ be the nilpotent cone in $M_n(F)$, i.e., the variety of all nilpotent matrices in this algebra. Denote by $\mathcal{S}_n$ the subspace of $M_n(F)$ consisting of all skew-symmetric bidiagonal matrices, by $\mathcal{T}_n$ the affine subspace of $M_n(F)$ consisting of all tridiagonal matrices $A = [a_{ij}]$ with $a_{i+1,i} = 1$ for $i = 1, \ldots, n-1$, and by $\mathcal{B}_n$ the affine subspace of $\mathcal{T}_n$ consisting of the matrices having zero diagonal. Finally, let

$$\mathcal{S}_n^* \subseteq \mathcal{S}_n, \quad \mathcal{T}_n^* \subseteq \mathcal{T}_n, \quad \mathcal{B}_n^* \subseteq \mathcal{B}_n$$

be the open subvarieties consisting of the matrices having nonzero entries along the first super-diagonal. Note that any matrix in $\mathcal{S}_n^*$ is similar to a unique matrix in $\mathcal{B}_n^*$, and the resulting map $\mathcal{S}_n^* \to \mathcal{B}_n^*$ is a 2^{n-1}–fold covering.

Unless stated otherwise, we assume from now on that $n = 2s + 1$ is odd. We are interested in the intersections $\mathcal{B}_n \cap \mathcal{N}_n$ and $\mathcal{S}_n \cap \mathcal{N}_n$ and their coordinate rings. We will relate these to a closed subvariety $\mathcal{V}_s \subseteq F^{n-1}$ to be defined shortly. We introduce first three maps

$$A_1 : F^{n-1} \to \mathcal{T}_{s+1}, \quad A_2 : F^{n-1} \to \mathcal{T}_s, \quad B : F^{n-1} \to \mathcal{B}_n,$$

by defining

$$A_1(\mathbf{p}) = \begin{bmatrix} -p_1 & p_1 p_2 & & & & \\ 1 & -p_2 - p_3 & p_3 p_4 & & & \\ & 1 & -p_4 - p_5 & & & \\ & & & \ddots & p_{n-2} p_{n-1} \\ & & & 1 & -p_{n-1} \end{bmatrix},$$

$$A_2(\mathbf{p}) = \begin{bmatrix} -p_1 - p_2 & p_2 p_3 & & & & \\ 1 & -p_3 - p_4 & p_4 p_5 & & & \\ & 1 & -p_5 - p_6 & & & \\ & & & \ddots & p_{n-3} p_{n-2} \\ & & & 1 & -p_{n-2} - p_{n-1} \end{bmatrix},$$

and

$$B(\mathbf{p}) = \begin{bmatrix} 0 & -p_1 & & & \\ 1 & 0 & -p_2 & & \\ & 1 & 0 & & \\ & & & \ddots & -p_{n-1} \\ & & & 1 & 0 \end{bmatrix},$$

where $\mathbf{p} = (p_1, \ldots, p_{n-1}) \in F^{n-1}$.

Proposition 5.1. *The characteristic polynomials of $A_1(\mathbf{p})$, $A_2(\mathbf{p})$ and $B(\mathbf{p})$ are related as follows.*

$$\text{(5.1)} \qquad \det(t\, I_{s+1} - A_1(\mathbf{p})) \;=\; t \det(t\, I_s - A_2(\mathbf{p})),$$
$$\text{(5.2)} \qquad \det(t\, I_n - B(\mathbf{p})) \;=\; t \det(t^2\, I_s - A_2(\mathbf{p})).$$

Hence, if one of the matrices $A_1(\mathbf{p})$, $A_2(\mathbf{p})$, $B(\mathbf{p})$ is nilpotent, so are the other two.

Proof. Following Givental [3], we define the matrices (of size $s + 1$)

$$U(\mathbf{p}) = \begin{bmatrix} -p_1 & & & & \\ 1 & -p_3 & & & \\ & 1 & -p_5 & & \\ & & & \ddots & \\ & & & & -p_{n-2} \\ & & & & 1 & 0 \end{bmatrix}$$

and

$$V(\mathbf{p}) = \begin{bmatrix} 1 & -p_2 & & & & \\ & 1 & -p_4 & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & -p_{n-1} \\ & & & & & 1 \end{bmatrix}.$$

Then $U(\mathbf{p})V(\mathbf{p}) = A_1(\mathbf{p})$ and

$$V(\mathbf{p})U(\mathbf{p}) = \begin{bmatrix} A_2(\mathbf{p}) & 0 \\ w & 0 \end{bmatrix},$$

where w is a row vector. Consequently,

$$\det(t\,I_{s+1} - A_1(\mathbf{p})) = \det(t\,I_{s+1} - U(\mathbf{p})V(\mathbf{p}))$$
$$= \det(t\,I_{s+1} - V(\mathbf{p})U(\mathbf{p})) = t\det(t\,I_s - A_2(\mathbf{p})).$$

It is easy to verify that there is a permutation matrix $\Pi_n \in M_n(F)$ such that

$$\Pi_n B(\mathbf{p})^2 \Pi_n^{-1} = \begin{bmatrix} A_1(\mathbf{p}) & 0 \\ 0 & A_2(\mathbf{p}) \end{bmatrix}$$

holds for all $\mathbf{p} \in F^{n-1}$. So, by using (5.1),

$$\det(t\,I_n - B(\mathbf{p})^2) = \det(t\,I_{s+1} - A_1(\mathbf{p}))\det(t\,I_s - A_2(\mathbf{p}))$$
$$= t\det(t\,I_s - A_2(\mathbf{p}))^2.$$

On the other hand it is easy to see that $\det(t\,I_n - B(\mathbf{p})) = \det(t\,I_n + B(\mathbf{p}))$ and so

$$\det(t\,I_n - B(\mathbf{p}))^2 = \det(t^2\,I_n - B(\mathbf{p})^2) = t^2\det(t^2\,I_s - A_2(\mathbf{p}))^2.$$

Therefore $\det(t\,I_n - B(\mathbf{p})) = \pm t\det(t^2\,I_s - A_2(\mathbf{p}))$ which after comparing the leading coefficients implies (5.2). □

The non-constant coefficients of the characteristic polynomial of $B(\mathbf{p})$, or equivalently of $A_1(\mathbf{p})$ or $A_2(\mathbf{p})$ define a variety (as we will see, reduced) inside F^{2s} which we denote by $\mathcal{V}_s$. In other words,

$$\mathcal{V}_s = A_1^{-1}(\mathcal{T}_{s+1} \cap \mathcal{N}_{s+1}) = A_2^{-1}(\mathcal{T}_s \cap \mathcal{N}_s) = B^{-1}(\mathcal{B}_n \cap \mathcal{N}_n).$$

Define also $\mathcal{V}_s^* = \mathcal{V}_s \cap (F^*)^{n-1}$ and note that the map B induces isomorphisms $\mathcal{V}_s \to \mathcal{B}_n \cap \mathcal{N}_n$ and $\mathcal{V}_s^* \to \mathcal{B}_n^* \cap \mathcal{N}_n$.

Let $\alpha_1 : \mathcal{V}_s \to \mathcal{T}_{s+1} \cap \mathcal{N}_{s+1}$ and $\alpha_1^* : \mathcal{V}_s^* \to \mathcal{T}_{s+1}^* \cap \mathcal{N}_{s+1}$ be the maps induced by A_1, and define similarly the maps α_2 and α_2^*.

Proposition 5.2. *$\mathcal{V}_s^*$ is a smooth, irreducible variety of dimension s and α_1^* is an open inclusion.*

Proof. By Kostant's work on the Toda lattice (see [4, Theorem 2.5]) the scheme-theoretic intersection $\mathcal{T}_{s+1}^* \cap \mathcal{N}_{s+1}$ defines a smooth, irreducible variety of dimension

s. In fact it is isomorphic to an open subset of F^s. It suffices, therefore, to show that α_1^* is an open embedding. A matrix

$$\begin{bmatrix} -a_1 & b_1 & & & \\ 1 & -a_2 & b_2 & & \\ & 1 & \ddots & & \\ & & & -a_s & b_s \\ & & & 1 & -a_{s+1} \end{bmatrix} \in \mathcal{T}_{s+1}^* \cap \mathcal{N}_{s+1}$$

lies in the image of α_1^* precisely if the denominators in the continued fraction expansions

$$p_1 = a_1, \quad p_2 = \frac{b_1}{a_1}, \quad p_3 = a_2 + \frac{b_1}{a_1}, \quad p_4 = \frac{b_2}{a_2 + \frac{b_1}{a_1}}, \quad p_5 = a_3 + \frac{b_2}{a_2 + \frac{b_1}{a_1}}$$

up to

$$p_{n-1} = \frac{b_s}{a_s + \frac{b_{s-1}}{\cdots + \frac{b_1}{a_1}}}$$

are all nonzero. This clearly defines an open subset of $\mathcal{T}_{s+1}^* \cap \mathcal{N}_{s+1}$. The above formulas for the p_i define an algebraic inverse from this open set to $\mathcal{V}_s^*$. $\qquad\square$

We have therefore proved that $\mathcal{B}_n^* \cap \mathcal{N}_n$ is a smooth, irreducible variety of dimension s.

Denote by Λ_s the set of subsequences $\lambda = (\lambda_1, \dots, \lambda_k)$ of $(1, 2, \dots, 2s)$ with k even, $0 \le k \le 2s$, and such that $\lambda_i \equiv i \pmod 2$ for each i. Let $\mathbf{p} = (p_1, \dots, p_{2s}) \in \mathcal{V}_s$ and let $\lambda_{\mathbf{p}} = (\lambda_1, \dots, \lambda_k)$ be the increasing sequence of indices such that

$$p_{\lambda_1} = \cdots = p_{\lambda_k} = 0$$

and all other coordinates p_i are nonzero. Since nilpotent matrices have zero determinant, it is easy to see that $\lambda_{\mathbf{p}} \in \Lambda_s$. Consequently, we have a map $\sigma : \mathcal{V}_s \to \Lambda_s$ defined by $\sigma(\mathbf{p}) = \lambda_{\mathbf{p}}$. We denote by $\mathcal{V}_s^\lambda$ the fibre of σ lying over the point $\lambda \in \Lambda_s$. This gives a set-theoretic partition

$$(5.3) \qquad\qquad \mathcal{V}_s = \coprod_{\lambda \in \Lambda_s} \mathcal{V}_s^\lambda.$$

These fibres are in fact smooth subvarieties. For instance, for the empty sequence $\emptyset \in \Lambda_s$ we have $\mathcal{V}_s^\emptyset = \mathcal{V}_s^*$. Moreover, we have the following description of the fibres.

Proposition 5.3. *For $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda_s$, $k = 2r$, we set $\lambda_0 = 0$, $\lambda_{k+1} = n$ and $s_i = (\lambda_i - \lambda_{i-1} - 1)/2$, $1 \le i \le k+1$. Then*

$$\mathcal{V}_s^\lambda \cong \prod_{1 \le i \le k+1} \mathcal{V}_{s_i}^*,$$

where, by convention, $\mathcal{V}_0^*$ is the variety consisting of a single point. In particular $\mathcal{V}_s^\lambda$ is a smooth variety of dimension $s - r$.

Proof. Let $\mathbf{p} \in \mathcal{V}_s^\lambda$. Then the characteristic polynomial of $B(\mathbf{p})$ agrees with the characteristic polynomial of the block matrix

$$M = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_{k+1}, \end{bmatrix},$$

where

$$M_i = \begin{bmatrix} 0 & -p_{\lambda_{i-1}+1} & & & \\ 1 & 0 & -p_{\lambda_{i-1}+2} & & \\ & 1 & 0 & & \\ & & & \ddots & -p_{\lambda_i-1} \\ & & & 1 & 0 \end{bmatrix}.$$

The matrix M is nilpotent if and only if each of the M_i is nilpotent. Therefore

$$\mathbf{p}_i := (p_{\lambda_{i-1}+1}, p_{\lambda_{i-1}+2}, \dots, p_{\lambda_i-1})$$

lies in $\mathcal{V}^*_{s_i}$. The resulting map

$$\begin{aligned} \mathcal{V}^\lambda_s &\rightarrow \prod_{1 \le i \le k+1} \mathcal{V}^*_{s_i} \\ \mathbf{p} &\mapsto (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{k+1}) \end{aligned}$$

is clearly an isomorphism.   □

Theorem 5.4. $\mathcal{V}_s$ *is an irreducible variety.*

Proof. $\mathcal{V}_s$ is defined as the intersection of s hypersurfaces, the zero sets of the coefficients of t^k, $k = 0, \dots, s-1$, of the characteristic polynomial of $A_2(\mathbf{p})$. Consequently, each irreducible component of $\mathcal{V}_s$ has dimension at least s. On the other hand, Proposition 5.3 implies that these dimensions are at most equal to s. Hence, $\mathcal{V}_s$ is an equidimensional variety of dimension s. It follows that $\mathcal{V}_s$ is irreducible.   □

Corollary 5.5. $\mathcal{S}_n \cap \mathcal{N}_n$ *and* $\mathcal{S}^*_n \cap \mathcal{N}_n$ *are varieties of pure dimension s. Moreover* $\mathcal{S}^*_n \cap \mathcal{N}_n$ *is smooth and an open dense subset of* $\mathcal{S}_n \cap \mathcal{N}_n$.

Proof. We have a well-defined map $\phi : \mathcal{S}_n \cap \mathcal{N}_n \rightarrow \mathcal{V}_s$ which takes a nilpotent skew-symmetric bidiagonal matrix with entries $(a_1, \dots, a_{n-1})$ along the upper diagonal, to an element $(-a_1^2, \dots, -a_{n-1}^2) \in \mathcal{V}_s$. This map restricts to a covering over each of the fibers $\mathcal{V}^\lambda_s$, which implies that $\mathcal{S}_n \cap \mathcal{N}_n$ is reduced and any irreducible component has dimension at most s. However, as for $\mathcal{V}_s$, any irreducible component of $\mathcal{S}_n \cap \mathcal{N}_n$ has dimension at least s. Therefore $\mathcal{S}_n \cap \mathcal{N}_n$ is of pure dimension s. As a consequence the s-dimensional fiber, $\mathcal{S}^*_n \cap \mathcal{N}_n$, is an open dense subset of $\mathcal{S}_n \cap \mathcal{N}_n$. Moreover, since $\mathcal{S}^*_n \cap \mathcal{N}_n$ is a covering space of $\mathcal{V}^*_s$, it is a smooth variety.   □

The problem of finding "nice" bidiagonal normal forms for orthogonal similarity classes of skew-symmetric matrices is identical to exhibiting "nice" representatives of the intersection $\mathcal{S}_n \cap \mathcal{N}_n$. In connection with this it is important to determine precisely the irreducible components of $\mathcal{S}_n \cap \mathcal{N}_n$. If $n = 3$ it is easy to check directly that $\mathcal{S}_3 \cap \mathcal{N}_3$ has two irreducible components.

Problem 4 Prove or disprove the following assertion. For $n > 3$ odd, the variety $\mathcal{S}^*_n \cap \mathcal{N}_n$ is connected, and $\mathcal{S}_n \cap \mathcal{N}_n$ is irreducible.

Remark 5.6. The above considerations are for varieties of $n \times n$ matrices where n is odd. For even $n = 2s$ we remark that $\mathcal{B}_{2s} \cap \mathcal{N}_{2s}$ is a variety of pure dimension $s - 1$, but no longer irreducible if $s > 1$. Namely it is easy to see that one has a decomposition into s components,

$$\mathcal{B}_{2s} \cap \mathcal{N}_{2s} = \bigcup_{j=1}^{s} \mathcal{I}_j,$$

where $\mathcal{I}_j$ is defined by the vanishing of the $(2j-1)^{st}$ entry on the upper diagonal. Each of these components is isomorphic to a product, $\mathcal{I}_j \cong \mathcal{V}_{j-1} \times \mathcal{V}_{s-j}$, and hence is irreducible and $(s-1)$-dimensional.

For the even-dimensional skew-symmetric case the scheme-theoretic intersection $\mathcal{S}_{2s} \cap \mathcal{N}_{2s}$ is not reduced, so we choose to consider $\mathcal{S}_{2s} \cap \mathcal{N}_{2s}$ as an intersection of algebraic sets. The resulting variety then again decomposes into a union of s subvarieties $\mathcal{I}_j^{\mathrm{skew}}$, where the definition of $\mathcal{I}_j^{\mathrm{skew}}$ is analogous to that of $\mathcal{I}_j$ above. Moreover, the component $\mathcal{I}_j^{\mathrm{skew}}$ is again isomorphic to a product,

$$(5.4) \qquad (\mathcal{S}_{2j-1} \cap \mathcal{N}_{2j-1}) \times (\mathcal{S}_{2s-2j+1} \cap \mathcal{N}_{2s-2j+1}),$$

and Corollary 5.5 implies that it has pure dimension $s-1$. Hence $\mathcal{S}_{2s} \cap \mathcal{N}_{2s}$ has pure dimension $s-1$.

The number of irreducible components of $\mathcal{S}_{2s} \cap \mathcal{N}_{2s}$, when $s > 1$, depends on the numbers of irreducible components of the subvarieties (5.4). Namely, since $\mathcal{S}_3 \cap \mathcal{N}_3$ has two irreducible components and thus $\mathcal{I}_2^{\mathrm{skew}}$ and $\mathcal{I}_{s-1}^{\mathrm{skew}}$ have at least two irreducible components each, the variety $\mathcal{S}_{2s} \cap \mathcal{N}_{2s}$ must have at least $s+2$ irreducible components.

Explicitly we have the following. In the case $s=1$ the algebraic set $\mathcal{S}_2 \cap \mathcal{N}_2$ consists of a single point. For $s=2$ there are precisely $s+2=4$ irreducible components in $\mathcal{S}_4 \cap \mathcal{N}_4$, obtained as described above. The case $s=3$ is exceptional with 6 irreducible components. This is because the variety $\mathcal{I}_2^{\mathrm{skew}}$ is isomorphic to $(\mathcal{S}_3 \cap \mathcal{N}_3) \times (\mathcal{S}_3 \cap \mathcal{N}_3)$ which has 4 irreducible components, and it can be checked that $\mathcal{I}_1^{\mathrm{skew}}$ and $\mathcal{I}_3^{\mathrm{skew}}$ are irreducible. For $s \geq 4$, if the assertion in Problem 4 is true then the variety $\mathcal{S}_{2s} \cap \mathcal{N}_{2s}$ has precisely $s+2$ irreducible components.

The results in this section may be compared with the normalized tridiagonal and the tridiagonal symmetric cases.

In the normalized tridiagonal case $\mathcal{T}_m \cap \mathcal{N}_m$ was shown to be irreducible (of dimension $m-1$) by Kostant [5]. This variety is of particular interest as its coordinate ring has another interpretation as the quantum cohomology ring $qH^*(\mathrm{SL}_m/B, F)$ of the flag variety SL_m/B. In this context, the coordinate ring of our variety $\mathcal{B}_n \cap \mathcal{N}_n$ can also be interpreted as the quotient of the quantum cohomology ring $qH^*(\mathrm{SL}_n/B, F)$ by the ideal generated by the Chern classes of the tautological line bundles, $x_i = c_1(L_i)$. Thus Theorem 5.4 implies that

$$qH^*(\mathrm{SL}_n/B, F)/(x_1, \ldots, x_n)$$

is an integral domain (for odd n).

The variety of symmetric nilpotent $m \times m$ matrices is also equidimensional, as can be shown without much difficulty, and is irreducible precisely if its intersection with the regular nilpotent orbit (which is smooth) is connected.

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